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Topology of singularities in the A_1 phase of superfluid ^{3}He

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Abstract. The topological classification of point and line singularities in the A_1 phase of superfluid ^{3}He is studied in both strong and weak magnetic fields and in both large and small volumes. Particle-like solutions are also discussed briefly.

1. Introduction

Line singularities (disgyrations) in the A phase of superfluid ^{3}He with $\boldsymbol{l} = \hat{\boldsymbol{\rho}}$ or $\boldsymbol{l} = \hat{\boldsymbol{\phi}}$ were originally suggested by de Gennes (1973). More recently, point singularities in the A and B phases with the $\boldsymbol{l}$ or $\boldsymbol{n}$ vector in the radial direction $\hat{\boldsymbol{r}}$ ('hedgehogs') have been suggested by Anderson and Brinkman (1975), Blaha (1976), Hu *et al* (1976) and Maki (1977). Topological discussions of these A- and B-phase singularities have been given by Volovik and Mineev (1977a, b), Toulouse and Kléman (1976) and Cross and Brinkman (1977), so that it is possible to tell which textures are topologically equivalent and which are forbidden to decay into one another by a topological conservation law. However, this kind of argument does not enable us to know which textures exist as local minima of the free energy. A study of which of these singularities exist as local minima of the free energy has been made by Bailin and Love (1978) for the A, B and A_1 phases, in the case of bulk superfluid (dimensions $\gg 10^{-3}$ cm).

We would like to emphasise that whereas the topological arguments can tell us, for example, whether the A phase 'hedgehog' $\boldsymbol{l} = \hat{\boldsymbol{r}}$ is a *true* point singularity (which it is not) they can *not* tell us whether it exists or not. This sort of argument can however provide us with other kinds of information not obtainable from the more detailed minimisation of the free energy. For instance, it can tell us whether two singularities can annihilate each other to produce the uniform texture, and which textures are prevented from decaying into another texture by a topological conservation law, and which only by a potential barrier. In the present paper we extend the topological arguments of earlier authors to the A_1 phase.

In § 2, the topological properties of A_1 phase singularities are discussed for the case of small volumes of superfluid (negligible dipolar interactions) in small and large magnetic fields. In § 3, a similar discussion is given for the case of bulk superfluid (important dipolar interactions) in small and large magnetic fields, and particle-like textures are also discussed. Section 4 contains our conclusions.

2. Topology of singularities in small volumes of superfluid

The application of topological ideas to the study of textural singularities has been described by Toulouse and Kléman (1976), by Cross and Brinkman (1977) and, in great detail, by Volovik and Mineev (1977a, b).

Three steps are involved. First, it is necessary to discover the topological space of order parameters under discussion. Second, the topologically inequivalent paths and surfaces traced out in order parameter space, as we go round a circle or over the surface of a sphere in real space, have to be studied, to classify line and point singularities, respectively. To do this the first and second homotopy groups for mappings from real space to order parameter space have to be calculated. Third, the topological quantum numbers of typical singularities have to be identified.

In the A_1 phase of superfluid ^{3}He the order parameter is of the form (see, e.g., Leggett 1975)

$$A_{\mu i} = \Delta d_\mu \Delta_i \tag{2.1}$$

where Δ is the magnitude of the order parameter and is fixed by the bulk term in the free energy,

$$\boldsymbol{\Delta} = \frac{1}{\sqrt{2}}(\boldsymbol{\alpha}_1 + \mathrm{i}\boldsymbol{\alpha}_2) \tag{2.2}$$

is a vector in ordinary space with $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ real mutually orthogonal unit vectors related to the $\boldsymbol{l}$ vector by $\boldsymbol{l} = \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$, and

$$\boldsymbol{d} = \frac{1}{\sqrt{2}}(\boldsymbol{\beta}_1 + \mathrm{i}\boldsymbol{\beta}_2) \tag{2.3}$$

is a spin-space vector with $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ real mutually orthogonal unit vectors. Since we consider samples of superfluid with small dimensions ($\ll 10^{-3}$ cm) in this section, we may neglect the orienting effect of dipole–dipole interactions. This effect is studied in § 3.

Consider first the case where the magnetic field is also negligible. (This is a very difficult case to realise experimentally, since the A_1 phase does not exist in zero magnetic field and has a small width in temperature in small magnetic fields.) Then, the general order parameter is obtained by independently choosing an orientation for the triads of axes ($\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$) and ($\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, $\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2$). This may be done by rotating each triad from a starting position ($\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$), and at first sight the space of allowed order parameters is that of SO(3)×SO(3), since we may write

$$\sqrt{2}\boldsymbol{d} = R_1(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}) \qquad \sqrt{2}\boldsymbol{\Delta} = R_2(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}),$$

where R_1, R_2 are arbitrary independent rotations. However, this involves some double counting since the order parameter $A_{\mu i}$ is the product of d_μ and Δ_i and $d_\mu\Delta_i$ is the same as $(\mathrm{e}^{-\mathrm{i}\alpha}d_\mu)(\mathrm{e}^{\mathrm{i}\alpha}\Delta_i)$ for any real α. Now

$$\sqrt{2}\mathrm{e}^{-\mathrm{i}\alpha}\boldsymbol{d} = R_1R^z(\alpha)(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})$$
$$\sqrt{2}\mathrm{e}^{\mathrm{i}\alpha}\boldsymbol{\Delta} = R_2R^z(-\alpha)(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}),$$

where $R^z(\alpha)$ is a rotation about the z axis by α. Thus the space of order parameters

$A_{\mu i}$ is the set R of all left cosets of the S_1 subgroup

$$S_1 = \{(R^z(\alpha), R^z(-\alpha)) \colon 0 \leq \alpha < 2\pi\}$$

of

$$SO(3) \times SO(3) = \{(R_1, R_2) \colon R_1, R_2 \in SO(3)\}.$$

Then we write

$$R = SO(3) \times SO(3)/S_1, \tag{2.4}$$

and since S_1 is a closed subgroup of the Lie group SO(3)×SO(3) it follows that SO(3)×SO(3) is a principal fibre bundle with group S_1. It follows that the homotopy groups $\pi_n(R)$ belong to an exact sequence of homomorphisms (see below), and may therefore be calculated, knowing the homotopy groups of S_1 and SO_3. (The method is described by Volovik and Mineev (1977a, b).)

Consider the slightly more general case

$$R = H/S_1 \tag{2.5}$$

where H is a principal fibre bundle with group S_1, and the homotopy groups of H are known (by using $\pi_n(A \times B) = \pi_n(A) + \pi_n(B)$ etc). This is the most general case we shall have to consider in the present work. The sequence of homomorphisms

$$\pi_2(S_1) \to \pi_2(H) \to \pi_2(R) \xrightarrow{\alpha} \pi_1(S_1) \xrightarrow{\beta} \pi_1(H) \xrightarrow{\gamma} \pi_1(R) \xrightarrow{\delta} \pi_0(S_1)$$

is exact, i.e. the image of any homotopy group in the sequence is the kernel of the subsequent mapping. Since $\pi_2(S_1) = \pi_0(S_1) = 0$ and $\pi_1(S_1) = Z$, and for all the cases we shall consider $\pi_2(H) = 0$, the sequence is

$$0 \to \pi_2(R) \xrightarrow{\alpha} Z \xrightarrow{\beta} \pi_1(H) \xrightarrow{\gamma} \pi_1(R) \xrightarrow{\delta} 0. \tag{2.6}$$

Information may be extracted from this sequence using the theorem that the image of any homotopy group in the sequence is the quotient group of that homotopy group with the kernel of the mapping, together with the definition of an exact sequence of homomorphisms. So

$$\pi_2(R) = \alpha\pi_2(R) \tag{2.7}$$

$$Z/\alpha\pi_2(R) = \beta Z \tag{2.8}$$

and

$$\pi_1(H)/\beta Z = \gamma\pi_1(H) = \pi_1(R) \tag{2.9}$$

(noticing that the kernel of the mapping δ is $\pi_1(R)$). In these equations the equality sign means isomorphism. Since, from (2.8), $\alpha\pi_2(R)$ is a subgroup of Z, the possibilities are

$$\alpha\pi_2(R) = 0,\ Z,\ \text{even integers, etc} \tag{2.10}$$

and correspondingly

$$\beta Z = Z, 0, Z_2, \text{etc} \tag{2.11}$$

and

$$\pi_1(R) = \pi_1(H)/Z,\ \pi_1(H)/0,\ \pi_1(H)/Z_2, \text{etc.} \tag{2.12}$$

To see which of these possibilities is actually realised we look at βZ. In the present case, the embedding of the S_1 subgroup in $H = \mathrm{SO(3)} \times \mathrm{SO(3)}$ induces the mapping $\beta\colon \pi_1(S_1) \to \pi_1(H)$. This latter group $\pi_1(H) = Z_2 + Z_2$, since $\pi_1(\mathrm{SO(3)}) = Z_2$. Now consider the closed path in $\mathrm{SO(3)} \times \mathrm{SO(3)}$ traced by the elements $(R^z(\alpha), R^z(-\alpha))$ of the S_1 subgroup as α covers the closed path 0 to 2π, n times. This is a representative of the element (n mod 2, n mod 2) of $\pi_1(H)$, so $\beta n = (n \bmod 2,\ n \bmod 2) \in Z_2 + Z_2$. Hence $\beta Z = Z_2$ in this case, essentially because *both* of the rotations $R^z(\alpha)$ and $R^z(-\alpha)$ are embedded in an SO(3). Consequently,

$$\pi_2(R) = \alpha\pi_2(R) = \text{even integers} \tag{2.13}$$

and

$$\pi_1(R) = (Z_2 + Z_2)/Z_2 = Z_2. \tag{2.14}$$

Equation (2.14) leads to only two types of line singularities (including the trivial one). Examples are the uniform texture and the simple vortex line:

$$\boldsymbol{\Delta} = \frac{1}{\sqrt{2}} \mathrm{e}^{\mathrm{i}\phi}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}), \qquad \boldsymbol{d} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}). \tag{2.15}$$

Two vortex lines can annihilate each other to produce the uniform texture. The straightforward $\boldsymbol{l} = \hat{\boldsymbol{\rho}}$ and $\boldsymbol{l} = \hat{\boldsymbol{\phi}}$ disgyrations, with $\boldsymbol{d}$ the uniform texture, are topologically equivalent to the simple vortex line.

Since $\pi_2(R) \neq 0$, there exist true point singularities in small samples of superfluid A_1 phase in small magnetic fields. Typical point singularities are given by

$$\sqrt{2}\boldsymbol{\Delta} = \cos\theta[\cos(m\phi)\hat{\boldsymbol{x}} + \sin(m\phi)\hat{\boldsymbol{y}}] - \sin\theta\hat{\boldsymbol{z}} + \mathrm{i}[-\sin(m\phi)\hat{\boldsymbol{x}} + \cos(m\phi)\hat{\boldsymbol{y}}] \tag{2.16}$$

and

$$\sqrt{2}\boldsymbol{d} = \cos\theta[\cos(n\phi)\hat{\boldsymbol{x}} + \sin(n\phi)\hat{\boldsymbol{y}}] - \sin\theta\hat{\boldsymbol{z}} + \mathrm{i}[-\sin(n\phi)\hat{\boldsymbol{x}} + \cos(n\phi)\hat{\boldsymbol{y}}] \tag{2.17}$$

with m and n integers. Not all such textures are true point singularities since some have a singular line on the z axis. $\boldsymbol{\Delta}$ has a singularity of the form $\mathrm{e}^{-\mathrm{i}m\phi}$ and $\boldsymbol{d}$ a singularity of the form $\mathrm{e}^{-\mathrm{i}n\phi}$ on the positive z axis so that the order parameter behaves as $\mathrm{e}^{-\mathrm{i}(m+n)\phi}$. Clearly, if we choose $m + n = 0$ the order parameter has no line singularity on either the positive or negative z axis. We must therefore restrict attention to $n = -m$. In accordance with equations (2.7) and (2.13), these point singularities are classified by an integer p which can be identified with the number of times a closed surface is covered in order parameter space when a sphere in a real space is covered once. For the examples given p is m. The *simplest* example with $p = 1$ is not that obtained by setting $m = -n = 1$ in equations (2.16) and (2.17), but rather

$$\sqrt{2}\boldsymbol{\Delta} = \hat{\boldsymbol{\theta}} + \mathrm{i}\hat{\boldsymbol{\phi}}, \qquad \sqrt{2}\boldsymbol{d} = \hat{\boldsymbol{\theta}} - \mathrm{i}\hat{\boldsymbol{\phi}}. \tag{2.18}$$

However, this is not a local minimum of the free energy and must relax into some more complicated texture with $p = 1$.

An alternative way of thinking about the topological quantum number p is as follows. Equations (2.16) and (2.17) with $n = -m$ define an element of $\pi_2(R)$ because they define a continuous mapping from a sphere in real space to a surface in R. They also define a continuous mapping from the (θ, ϕ) plane $(0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \phi \leqslant 2\pi)$ to $\mathrm{SO(3)} \times \mathrm{SO(3)}$ (though *not* from a sphere in real space to $\mathrm{SO(3)} \times \mathrm{SO(3)}$). If we consider the curve in $SO(3)\times SO(3)$ obtained by going round the boundary of the above region of the (θ, ϕ) plane, we see that it is an element of $\pi_1(S_1)$ with winding number $2m$. (There is winding number m from the part of the path with $\theta = 0$ and also from the part with $\theta = \pi$. The sides at constant ϕ cancel.) In this way, the element of $\pi_2(R)$ defined by equations (2.16) and (2.17) with $n = -m$ is mapped onto an element of $\pi_1(S_1)$. This mapping α is such that the element of $\pi_2(R)$ with $p = m$ is mapped onto the element of $\pi_1(S_1)$ with winding number $2m$. Consequently $\alpha\pi_2(R)$ is the even integers, in agreement with equation (2.13). The procedure just described provides an alternative way of discovering the topological quantum number p of a given texture in $\pi_2(R)$.

Turning now to the case of large magnetic field, say along the z axis, the space of allowed order parameters becomes

$$R = (SO(3)\times S_1)/S_1. \tag{2.19}$$

This is because $\mathbf{\Delta}$ is allowed all orientations as before, but $\boldsymbol{d}$ must have $\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2$ along the z axis. The S_1 in the denominator is defined as following equation (2.4). The general arguments of equations (2.5) to (2.12) still apply but with H now equal to $SO(3)\times S_1$. In this case, only one of the rotations involved in defining the S_1 in the denominator of equation (2.19) is now embedded in an SO(3). Thus the embedding induces the mapping of the element $n \in Z = \pi_1(S_1)$ onto $\beta n = (n \bmod 2, n) \in Z_2 + Z = \pi_1(SO(3)\times S_1)$. Hence $\beta Z = Z$. Consequently

$$\pi_2(R) = 0 \tag{2.20}$$

and

$$\pi_1(R) = (Z_2 + Z)/Z = Z_2. \tag{2.21}$$

Thus in large magnetic fields there are no true point singularities, and there are two types of line singularity, examples of which are the uniform texture and the simple vortex line given in (2.15). Two vortex lines can annihilate each other to produce a uniform texture. Disgyrations with $\boldsymbol{l} = \hat{\boldsymbol{\rho}}$ and $\boldsymbol{l} = \hat{\boldsymbol{\phi}}$ are topologically equivalent to the simple vortex line.

3. Topology of singularities in large volumes of superfluid

When large volumes of superfluid are considered (dimensions $\gg 10^{-3}$ cm) the orienting effect of dipole–dipole interactions is important. This causes the vector $\boldsymbol{l}$ to lie at right angles to $\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2$. (See for example Leggett (1975).)

Consider first the case of negligible magnetic field. Then, the most general allowed order parameter is obtained from the reference triads defined by

$$(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_1\times\boldsymbol{\alpha}_2) = (\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}})$$

and

$$(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_1\times\boldsymbol{\beta}_2) = (\hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{x}})$$

by a rotation of $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ about the z axis, a rotation of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ about the x axis, and finally a simultaneous identical rotation of the two resultant triads about a general axis. Thus

$$\sqrt{2}\,\boldsymbol{d} = R_1R^x(\beta)(\hat{y}+\mathrm{i}\hat{z})$$

and

$$\sqrt{2}\,\boldsymbol{\Delta} = R_1R^z(\gamma)(\hat{x}+\mathrm{i}\hat{y}),$$

where R_1 is the common rotation and $R^x(\beta)$, $R^z(\gamma)$ are the rotations about the x axis, z axis by β, γ. Thus at first sight the space of order parameters is

$$H = \{(R_1R^x(\beta), R_1R^z(\gamma)) : R_1 \in \mathrm{SO}(3), 0 \leq \beta, \gamma < 2\pi\}. \tag{3.1a}$$

As before, this over counts because $(\mathrm{e}^{-\mathrm{i}\alpha}d_\mu)(\mathrm{e}^{\mathrm{i}\alpha}\Delta_i) = d_\mu\Delta_i$ for any real α. So in this case the manifold R of the order parameter $A_{\mu i}$ is a *subset* of the set of left cosets of the subgroup

$$S_1 = \{(R^x(\alpha), R^z(-\alpha)) : 0 \leq \alpha < 2\pi\} \tag{3.1b}$$

in $\mathrm{SO}(3)\times\mathrm{SO}(3)$. Thus H is a sub-bundle of the principal bundle over the above S_1 group, and as before the homotopy groups $\pi_n(R)$ belong to an exact sequence of homomorphisms. They may therefore be calculated provided we know $\pi_n(H)$. But we can show that H is homeomorphic to $\mathrm{SO}(3)\times S_1\times S_1$. To see this consider the mapping $\theta:\mathrm{SO}(3)\times S_1\times S_1 \to H$ such that $\theta(R_1, \beta, \gamma) = (R_1R^x(\beta), R_1R^z(\gamma))$. θ is bijective, since if $R_1R^x(\beta) = R_1'R^x(\beta')$ and $R_1R^z(\gamma) = R_1'R^z(\gamma')$ then $R^z(\gamma'-\gamma) = R^x(\beta'-\beta)$, which implies $\gamma' = \gamma$, $\beta' = \beta$ and $R_1' = R_1$. Further, θ and θ^{-1} are plainly continuous, so θ is a homeomorphism.

So the general arguments of equations (2.5) to (2.12) can be applied with H equal to $\mathrm{SO}(3)\times S_1\times S_1$. Since neither of the rotations involved in defining the denominator S_1 is embedded in an SO(3), we have $\beta Z = Z$, with the result that

$$\pi_2(R) = 0 \tag{3.2}$$

and

$$\pi_1(R) = (Z_2 + Z + Z)/Z = Z_2 + Z. \tag{3.3}$$

There are thus no true point singularities and the line singularities are classified by a topological quantum number $N(Z_2)$, which can take the values 0 and 1, and a topological quantum number $N(Z)$, which can take all integral values. The former quantum number is associated with simultaneous rotations of the two reference triads about the same axis, and the latter quantum number is associated with the residual phase of the order parameter. Thus the order parameter given by

$$\sqrt{2}\,\boldsymbol{\Delta} = \mathrm{e}^{\mathrm{i}m\phi}[\cos(n\phi)\hat{x} + \sin(n\phi)\hat{z} + \mathrm{i}\hat{y}]$$
$$\sqrt{2}\,\boldsymbol{d} = \hat{y} + \mathrm{i}[-\sin(n\phi)\hat{x} + \cos(n\phi)\hat{z}]$$

is characterised by the pair $(n \bmod 2, m) \in Z_2 + Z$.

The examples of disgyrations in a small magnetic field given by Bailin and Love (1978, equations (4.12)–(4.15)) both have the topological quantum numbers $N(Z_2) = 1$, $N(Z) = 1$. Two such singularities can annihilate each other to produce the uniform texture.

Now consider the case of strong magnetic field, say along the z axis.

The vector $\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2$ is constrained to lie along the z axis, and $\boldsymbol{l}$ is constrained by the dipole–dipole interaction to lie perpendicular to $\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2$ i.e. in the x–y plane. Thus

$$\sqrt{2}\,\boldsymbol{d} = R^z(\beta)(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}}),$$
$$\sqrt{2}\,\boldsymbol{\Delta} = R^z(\delta)R^x(\gamma)(\hat{\boldsymbol{y}} + \mathrm{i}\hat{\boldsymbol{z}})$$

using the previous notation. In this case, then,

$$H = \{(R^z(\beta), R^z(\delta)R^x(\gamma)) : 0 \leqslant \beta, \gamma, \delta < 2\pi\} \tag{3.4a}$$

is a sub-bundle of the principle bundle with group

$$S_1 = \{(R^z(\alpha), R^x(-\alpha)) : 0 \leqslant \alpha < 2\pi\} \tag{3.4b}$$

and H is homeomorphic to $S_1 \times S_1 \times S_1$.

Again, the arguments of § 2 can be applied with H equal to $S_1 \times S_1 \times S_1$. Since neither of the rotations involved in defining the denominator S_1 is embedded in an SO(3), we must have $\beta Z = Z$, and the result is

$$\pi_2(R) = 0 \tag{3.5}$$

and

$$\pi_1(R) = (Z + Z + Z)/Z = Z + Z. \tag{3.6}$$

There are no true point singularities, and the line singularities are characterised by two integral quantum numbers associated with rotations of $\boldsymbol{l}$ about the z axis and the phase of the order parameter, respectively. Thus the order parameter given by

$$\sqrt{2}\,\boldsymbol{\Delta} = -\sin(n\phi)\hat{\boldsymbol{x}} + \cos(n\phi)\hat{\boldsymbol{y}} + \mathrm{i}\hat{\boldsymbol{z}}$$
$$\sqrt{2}\,\boldsymbol{d} = \mathrm{e}^{im\phi}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})$$

is characterised by the pair $(n, m) \in Z + Z$. The examples given by Bailin and Love (1978, equations (4.18) and (4.19)) both have topological quantum numbers $(n, m) = (1, 0)$.

For completeness, we also comment on particle-like textures where the order parameter approaches the same uniform texture as we go to infinity in any direction. Such structures are classified by the homotopy group $\pi_3(R)$. (See Volovik and Mineev (1977a, b) and also Shankar (1977) and Finkelstein (1966).)

In small magnetic fields, R is given by equation (3.1) and if we write $R = H/S_1$ and consider the exact sequence

$$\pi_3(S_1) \to \pi_3(H) \to \pi_3(R) \to \pi_2(S_1)$$

i.e.

$$0 \to Z \to \pi_3(R) \to 0,$$

we see that $\pi_3(R) = Z$. In this case, particle-like textures exist. On the other hand, in large magnetic fields, R is given by equation (3.4) and the corresponding exact sequence is

$$0 \to 0 \to \pi_3(R) \to 0$$

leading to $\pi_3(R) = 0$ and no particle-like textures. (The corresponding results for volumes of superfluid small enough to neglect the dipolar forces are $\pi_3(\mathrm{SO}(3) \times \mathrm{SO}(3)/S_1) = Z + Z$ and $\pi_3(\mathrm{SO}(3) \times S_1 \times S_1/S_1) = Z$.)

4. Conclusions and discussion

The topological discussion has shown that true point singularities in the A_1 phase can only exist in small volumes of superfluid and then negligible magnetic fields are required. The experimental conditions for observing such textures may not be realised except in the metastable A_1 phase produced by switching off the magnetic field. On the other hand a variety of line singularities exists in all conditions.

As far as domain walls are concerned the situation in the A_1 phase in large magnetic fields is topologically more similar to the B phase than the A phase, since domain walls in the A_1 phase are not 'held up' by any topological conservation law but only by a potential barrier. This is because $\pi_0((S_1 \times S_1 \times S_1)/S_1) = 0$. The detailed properties of domain walls in the A_1 phase will be presented in a forthcoming paper.

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